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The revision guide for Mathematics Basic Course (SF1659)

Roots

Vieta's formulae

Sum of roots: **-b/a** (the n-1 term divided by the nth term)
Product of roots: **k/a** (when n is even) and **-k/a** (when n is odd). The **k** is the constant term.

This is good when you have to guess the roots and the visual inspection does not work.

Discriminant

In the quadratic equation formula, the discriminant tells some things about the roots:

1. if discriminant > **0**, the equation has two real roots
2. if discriminant **= 0**, the equation has exactly one root
3. if discriminant **< 0**, the equation has not real roots

The discriminant can also be maximized in order to either figure out a specific condition when the distance between roots is as great as possible or as small as possible. If differentiation cannot be used, axis of symmetry, **-b/(2a)** can be used instead. NB: make sure to check whether it is a maxima or a minima.

$$x_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}, \qquad \text{where } \Delta \text{ is the descriminant}$$

Sequences

Arithmetic progression

An arithmetic progression is a sequence of numbers that follow the pattern below:

$$u_n = u_1 + (n-1)d$$

d – is the common difference

u_1 – is the first term.

The sum can be expressed as shown below:

$$S_n = \frac{n}{2}[2u_1 + (n-1)d]$$

or

$$S_n = \frac{n}{2}(u_1 + u_n)$$

NB: the latter version of the sum is sometimes easier to use. If you know the *first term*, *the last term* and the *common difference*, you can deduce the sum.

Geometric progression

A geometric progression is defined as shown below:

$$u_n = u_1 r^{n-1}$$

Where r is the common ratio which you can calculate by $r = \frac{a_{n+1}}{a_n}$.

The sum of a geometric progression is given by:

$$S_n = \frac{u_1(r^{n+1} - 1)}{r - 1}$$

NB: You have to be able to deduce this formula from scratch, i.e. it's not obvious. A good starting point is to multiply both sides by $(k-1)$ and you will soon realise that this is what you get.

Functions

Basics

D_f - definitionsmängd (domain of *f*)

V_f - värdemängd (range of *f*)

Invers (inverse)

A function that brings us back to the original value, i.e.

$$f^{-1} : f^{-1}(f(x)) = x, \quad \forall x \in D_f$$

The domain (D_f) of f is the range (V_f) of f^{-1} and vice versa.

Injektiv (one-to-one)

An injective function is reversible, i.e.

$$\forall x_1, x_2 \in D_f, \qquad x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$$

Using the logical equivalence,

$$A \Rightarrow B \qquad \Leftrightarrow \qquad \neg B \Rightarrow \neg A$$

The statement can be reversed, i.e.

$$\forall x_1, x_2 \in D_f, \qquad f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

Växande och avtagande (increasing and decreasing) - monoton

A function is **increasing (växande)** if,

$$\forall x_1, x_2 \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \leq f(x_2)$$

And **decreasing (avtagande)**

$$\forall x_1, x_2 \qquad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \geq f(x_2)$$

Both ***increasing (växande)*** *and* ***decreasing(avtagande)*** *functions are called* ***monoton****.*

Begränsad och obegränsad (bounded and unbounded)

A **bounded (begränsad)** function's values will not be unboundedly large. There are two types, **uppåt begränsad** and **nedåt begränsad**.

Uppåt begränsad

$$\text{if } \exists M: \qquad f(x) \quad \leq \quad M, \qquad \forall x \in D_f$$

or similarly **nedåt begränsad**,

$$\exists M: \qquad f(x) \quad \geq \quad M, \qquad \forall x \in D_f$$

Udda och jämna (even and odd)

A function can be even or odd. There are cases when it's neither even nor odd.

Even

$$f(-x) \quad = \quad f(x) \qquad \forall x \in D_f$$

Odd

$$f(-x) \quad = \quad -f(x) \qquad \forall x \in D_f$$

Trigonometric functions and identities

Identities

$$\cos(x + 2\pi k) = \cos x, \quad k \in \mathbb{Z}$$
$$\sin(x + 2\pi k) = \sin x, \quad k \in \mathbb{Z}$$
$$\cos^2 x + \sin^2 x = 1$$
$$\tan(x + \pi k) = \tan x, \quad k \in \mathbb{Z}$$

Because cos is **even** and sin is **odd**, it follows that,

$$\cos(-x) = \cos x$$
$$\sin(-x) = -\sin x$$
$$\tan(-x) = -\tan x$$

The unit circle provides us with some other neat identities,

$$\cos(x + \pi) = -\cos x$$
$$\sin(x + \pi) = -\sin x$$

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NB: The identities that we saw in the beginning of this section are quite useful in trigonometric equations. Say you have something like $\sin x = \sin 3x$ does not simply have the solution $x = 0$ but rather $x = \pi k \quad or \quad x = \frac{\pi}{4} + 2\pi k$.

Why two solutions? Well, in both sine and cosine we have the **ambiguous case** – a situation where these functions given two different values will result in the same sine/cosine value.

For sine, it's

$$\sin a = \sin b \quad \Leftrightarrow \quad a = b + 2\pi k \quad or \quad a = \pi - (b + 2\pi k)$$

For cosine,

$$\cos a = \cos b \quad \Leftrightarrow \quad a = \pm b + 2\pi k$$

Have this in mind when you remove the *sin* for instance. Let's return to the problem:

$$\begin{aligned} &\sin 3x = \sin x \\ &3x = x + 2\pi k \qquad or \qquad 3x = \pi - (x + 2\pi k) \\ &x = \pi k \qquad\qquad\qquad\quad x = \frac{\pi}{4} - \frac{\pi}{2} k \end{aligned}$$

Addition formulae for trig functions

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

Trigonometric inverses

Each trigonometric function, e.g. sin,cos,tan,cot,csc,sec have an inverse in a set interval only. For *sin*, an inverse exists if and only if the domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The range of will be $[-1,1]$

Consequently, the *arcsin* will have the domain $[-1,1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The *cosine* function has to have similar constraints to allow the existence of an inverse. The domain has to be $[0, \pi]$ and the range will be $[-1,1]$.

The *arccos* has the domain $[-1,1]$ and the range $[0, \pi]$. We basically interchange the domain and the range.

The *tangent* has to have the domain of $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and the range of all real numbers. The inverse, *arctan* has thus the domain of all real numbers and the range of $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Note that the endpoints are not included; it's an open interval.

Reciprocal identities

These are good to keep in mind, although some might appear less frequent.

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$$\csc x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{1}{\tan x}$$

Trig functions and their inverses combined

Sometimes we might get expressions as $\arccos(\cos x)$ *or* $\tan(\arctan x)$ and you might already see that it's a good idea to consider the restrictions we've made to the trig functions.

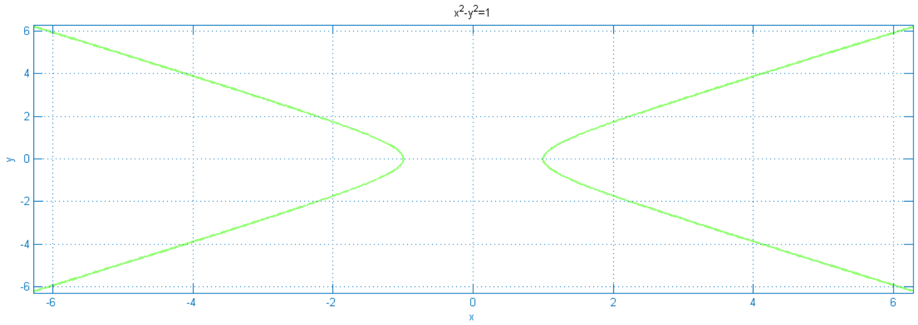
Generally,

$$f(f^{-1}(y)) = y, \qquad \forall y \in D_{f^{-1}} = V_f$$
$$f^{-1}(f(x)) = x \qquad \forall x \in D_f$$

It might be hard to remember all that, but it is usually good to keep in mind the purpose of inverses. In the first case, it's equal to y for all y in the range of the function. Why? Because we want the inverse to map back to the original value put in into the function.

Hyperbolic functions

Instead of thinking about the unit circle, we imagine a hyperbola.



$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad x \in \mathbb{R}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad x \in \mathbb{R}$$

The Pythagorean identity with hyperbolic trig functions is:

$$\cosh^2 x - \sinh^2 x = 1$$

This can be easily proved.

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Sums

Binomial sums

$$\sum_{0\le k\le n}\binom{k}{m}=\binom{0}{m}+\binom{1}{m}+\ldots+\binom{n}{m}=\binom{n+1}{m+1},\qquad \text{integers } m,n\ge 0$$

$$\sum\binom{x}{m}\delta x=\binom{x}{m+1}+C$$

NB: the constant is not necessarily a number. However, by taking the difference, it will disappear. Recall definite and indefinite integrals.

Binomial coefficients

$$\binom{r}{m}\binom{m}{k}=\binom{r}{k}\binom{r-k}{m-k}$$

Absorption identity

$$\binom{r}{k}=\frac{r}{k}\binom{r-1}{k-1},\qquad \text{integers } k\ne 0$$

Row identity

$$2^n=\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}$$

Miscellaneous

Log trick

EX: Use $e^{a+b}=e^a e^b$ to prove that $\ln(ab)=\ln a+\ln b, \qquad a,b>0$.

There is a hidden premise here, i.e. $e^x=y \quad\Leftrightarrow\quad \ln y=x$.

So, take log of both sides:

$$\ln(e^{a+b})=\ln(e^a e^b)$$
$$(a+b)\ln e=a+b$$
$$\ln e^a+\ln e^b=a\ln e+b\ln e=a+b$$

QED.

Questions or suggestions?

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