Bases and Transformations

Artem Los
(arteml@kth.se)

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Bases and Coordinates

What is a Base?

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Vector $[\vec{a}]_B$ is a *linear combination* of the base vectors $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$, i.e. $\vec{a} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_k\vec{v_k}$

How do we find the coefficients?

We know that $\vec{a} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$ can be written as:

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Solve the system of linear equations to find $\vec{c}$.

Example

Problem. Given that $B = \{(1,0,-1),(1,-1,0)\}$ find the coordinates of $\vec{v} = (1,-2,1)$ with respect to the basis B, i.e. find $[\vec{v}]_B$.

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$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -2 \\ -1 & 0 & 1 \end{array}\right) \sim \text{el. row op.} \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

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$\therefore [\vec{v}]_B = (-1,2)$, *which is coordinates of* $\vec{v}$ *in* B.

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We can always verify that $[\vec{v}]_B$ really represents $\vec{v}$ in B. Recall the matrix with the basis vectors as its columns, i.e.

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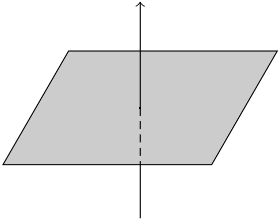
Now, multiply B with the coefficients we got to verify $B[\vec{v}]_B = \vec{v}$

$$B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Representing a base



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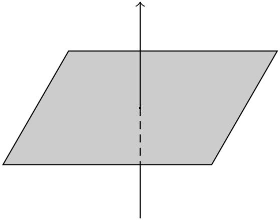

- A base can represent a space that is a *line*, a *plane*, a *hyperplane*, etc.
- We can tell this by looking at the *dimension of the base* - the *number* of base vectors.

Standard Matrix

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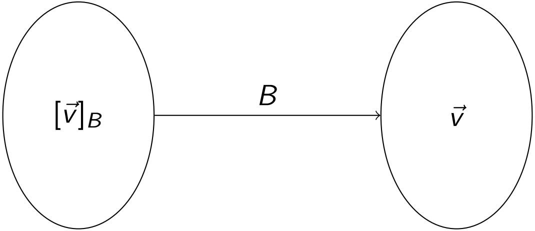

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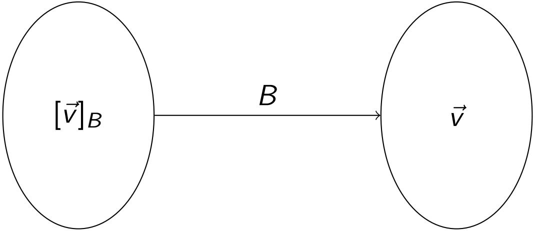

Problem. Given $\vec{v}$, how do we find $[\vec{v}]_B$?
Hint. Recall that $B[\vec{v}]_B = \vec{v}$.

From standard base to base B

Think of B as a **transformation matrix** of $B : \mathbb{R}^n \to \mathbb{R}^n$.

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B^{-1} is a linear transformation that **converts** all points in the **standard basis** to base B.

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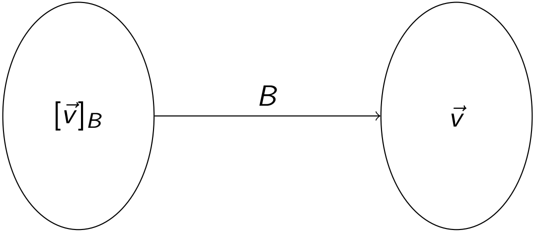

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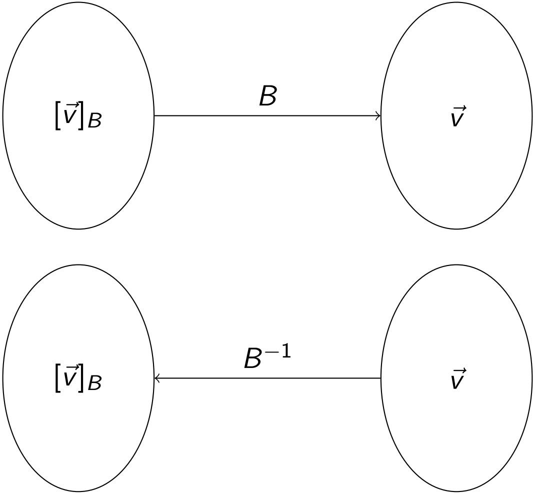

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Problem. Find the matrix that converts a point in base B to the standard matrix and vice versa, given that
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$$\begin{pmatrix} 0.5 & 0.5 & 1 \\ -0.75 & 0.75 & -1 \\ 0.25 & -0.25 & 0 \end{pmatrix}$$

Special Notation

Common ways of representing B

Instead of using the notations B and B^{-1} (not that meaningful), we can refer to them (in our example) as:

$$B = {}_S T_B = T_{B \to S} \qquad \text{from B to S}$$

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Using ${}_S T_B$, we can think that the **subscripts** specify what is **allowed** to be on the **left** and on the **right** of it, so this is allowed:

$${}_S T_B [\vec{v}]_B$$

whereas the case below is not:

$${}_S T_B \vec{v}$$

Multiple Transformations

Combining transformations

Problem. Suppose that we are given a linear transformation $Rot : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates a point by 90°. However, we want it to rotate points in base B but Rot **only accepts points in the standard basis**. Find the **transformation matrix** that rotates points in base B.

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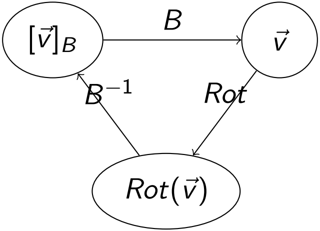

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Given a point $[\vec{v}]_B$, we get:

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By plugging in numbers:

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$$= \begin{pmatrix} 0 & -1 & -1 \\ 1.5 & 2.5 & 7.5 \\ -0.5 & -0.5 & -1.5 \end{pmatrix}$$

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$\therefore$ We now have a matrix that will rotate points 90° in base B.