Bases and Transformations

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January 30th, 2017

Bases and Coordinates

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2. Standard Matrix
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What is a Base?

What is a base?

A **base** allows us to describe coordinates from a different frame of reference

$$B = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$$

$$[\vec{a}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

Vector $[\vec{a}]_B$ is a *linear combination* of the base vectors $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$, i.e. $\vec{a} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$

How do we find the coefficients?

We know that $\vec{a} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$ can be written as:

$$\begin{pmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \dots & \vec{v_k} \\ | & | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \vec{a}$$

In other words,

$$B\vec{c} = \vec{a}$$

Solve the system of linear equations to find $\vec{c}$.

Example

Problem. Given that $B = \{(1, 0, -1), (1, -1, 0)\}$ find the coordinates of $\vec{v} = (1, -2, 1)$ with respect to the basis B, i.e. find $[\vec{v}]_B$.

Step 1: Find coefficients c_1, c_2 such that

$$c_1(1, 0, -1) + c_2(1, -1, 0) = \overbrace{(1, 2, -1)}^{\vec{v}}.$$

Step 2: Solve the system of linear equations

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -2 \\ -1 & 0 & 1 \end{array}\right) \sim \text{el. row op.} \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

$\therefore [\vec{v}]_B = (-1, 2)$, *which is coordinates of* $\vec{v}$ *in* B.

Example

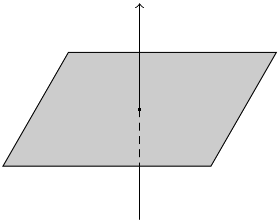
We can always verify that $[\vec{v}]_B$ really represents $\vec{v}$ in B. Recall the matrix with the basis vectors as its columns, i.e.

$$B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Now, multiply B with the coefficients we got to verify $B[\vec{v}]_B = \vec{v}$

$$B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

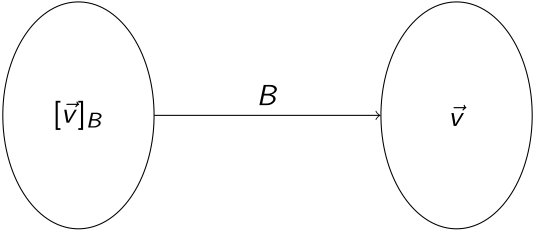
Representing a base



- A base can represent a space that is a *line*, a *plane*, a *hyperplane*, etc.
- We can tell this by looking at the *dimension of the base* - the *number* of base vectors.

Standard Matrix

Standard matrix



Problem. Given $\vec{v}$, how do we find $[\vec{v}]_B$?
Hint. Recall that $B[\vec{v}]_B = \vec{v}$.

From standard base to base B

Think of B as a **transformation matrix** of $B : \mathbb{R}^n \to \mathbb{R}^n$.

$$B[\vec{v}]_B = \vec{v}$$

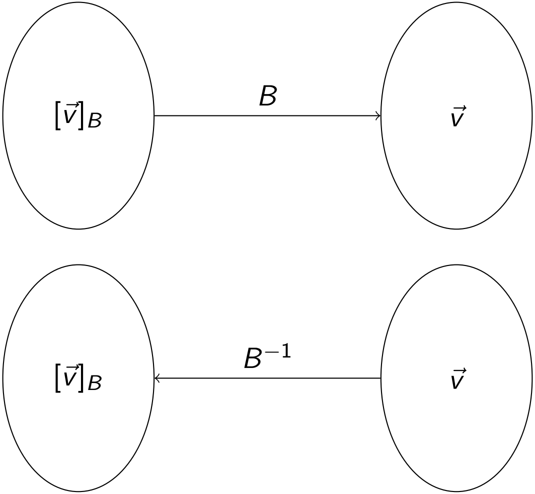
To solve for $[\vec{v}]_B$, multiply by the **inverse**.

$$B^{-1}[\vec{v}]_B = B^{-1}\vec{v}$$

$$[\vec{v}]_B = B^{-1}\vec{v}$$

B^{-1} is a linear transformation that **converts** all points in the **standard basis** to base B.

Standard matrix



Example

Problem. Find the matrix that converts a point in base B to the standard matrix and vice versa, given that $B = \{(1,1,0),(1,1,-1),(5,1,-3)\}$.

Step 1: We want to **find** B **matrix**, which is found by using the **base vectors** as **column vectors**, i.e.

$$\begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix}$$

Step 2: To convert points from the standard basis to base B, we need to find B^{-1}

$$\begin{pmatrix} 0.5 & 0.5 & 1 \\ -0.75 & 0.75 & -1 \\ 0.25 & -0.25 & 0 \end{pmatrix}$$

Special Notation

Common ways of representing B

Instead of using the notations B and B^{-1} (not that meaningful), we can refer to them (in our example) as:

$$B = {}_S T_B = T_{B \to S} \qquad \text{from B to S}$$

$$B^{-1} = {}_S T_B = T_{S \to B} \qquad \text{from S to B}$$

Using ${}_S T_B$, we can think that the **subscripts** specify what is **allowed** to be on the **left** and on the **right** of it, so this is allowed:

$${}_S T_B [\vec{v}]_B$$

whereas the case below is not:

$${}_S T_B \vec{v}$$

Multiple Transformations

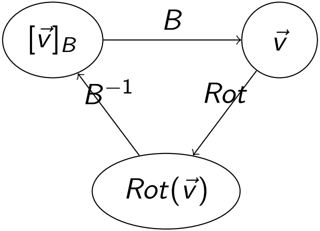
Combining transformations

Problem. Suppose that we are given a linear transformation $Rot : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates a point by 90°. However, we want it to rotate points in base B but Rot **only accepts points in the standard basis**. Find the **transformation matrix** that rotates points in base B.

Step 1: Convert the point $[\vec{v}]_B$ (in base B) to $\vec{v}$ (in standard basis)

Step 2: Perform the rotation Rot on $\vec{v}$

Step 3: Convert $Rot(\vec{v})$ to base B, i.e. $[Rot(\vec{v})]_B$



Example

Given a point $[\vec{v}]_B$, we get:

$$\underbrace{B^{-1} \times Rot \times B \times [\vec{v}]_B}_{\text{always from \textbf{right} to \textbf{left}}}$$

By plugging in numbers:

$$\begin{pmatrix} 0.5 & 0.5 & 1 \\ -0.75 & 0.75 & -1 \\ 0.25 & -0.25 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 & -1 \\ 1.5 & 2.5 & 7.5 \\ -0.5 & -0.5 & -1.5 \end{pmatrix}$$

$\therefore$ We now have a matrix that will rotate points 90° in base B.