Spectral Theorem and Quadratic Forms

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Overview

Symmetric Matrices

Definition

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As we will see later, **quadratic form** is one of the ways of exploiting **diagonalization symmetric matrices**.

Theorems

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- For any other case, we can always use Gram-Schmidt method to find an orthonormal basis.

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$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Quadratic Form

Definition

Motivation. Sometimes, we have equations of shapes that are transformed in some way (eg. rotated ellipse). However, we want to be able to deduce certain information (eg. major/minor of an ellipse). Then, we want to change the coordinate system in such a way that the ellipse is depicted in the 'normal form', with centre in the origin.

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The point is, we want to diagonalize Q to get rid of non-quadratic terms, i.e. here we want $b = 0$.

Classification of quadratic forms

- $Q(\vec{x})$ is positive definite if $Q(\vec{x}) > 0 \qquad \forall x \neq 0$
- $Q(\vec{x})$ is positive semi-definite if $Q(\vec{x}) \geq 0 \qquad \forall x$
- $Q(\vec{x})$ is positive indefinite if $Q(\vec{x}) > 0$ for some $\vec{x}$ and $Q(\vec{x}) < 0$ for some $\vec{x}$

Other forms (eg. negative definite) follow from these definitions.

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$$\therefore Q(\vec{y}) = 2.5y_1^2 - 0.5y_2^2$$

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Task: Express $(3, 2)_X$ in Y coordinates

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Since for all $\vec{y} \neq 0$, $Q(\vec{y}) = 6y_1^2 + 2y_2^2 > 0$, A is positive definite.

Recommended reading (review)

- Eigenvalues
- Diagonalization (general technique)