Determinants

Artem Los
(arteml@kth.se)

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Overview

1. What is a Determinant?
2. Rules and Theorems
3. Area of a Parallelogram

What is a Determinant?

Definition

Determinant

In linear algebra, the determinant is a **useful value** that can be computed from the elements of a **square matrix**. (from https://en.wikipedia.org/wiki/Determinant)

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For example,

- Deducing if matrix A is invertible, i.e. $\exists A^{-1}$
- Finding area of a parallelogram
- Calculating **eigenvalues** (next week)

Computing determinant

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Increasing dimension. 3×3

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Question. In the example below, why is a positive but b negative?

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In general, the coefficient is

$$(-1)^{i+j}$$

for element in ith row and jth column

Example

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Step 1: "Golden rule of determinant calculations: be lazy". Pick the row/column with most zeros. Why?

$$A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{6} \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 6(3-2) = 6$$

Rules and Theorems

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Goal. Get as many zeros as possible, for then we get less terms to compute.

Rules (continued)

Constant term factorization. $\det B = c \det A$

$$\begin{vmatrix} c*1 & c*2 & c*3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = c \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

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Adding two rows/columns. $\det B = \det A$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \{R_2 : R_2 - R_1\} = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 0 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

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Theorem. Given that A is a square matrix, i.e. $n \times n$, the following statements are equivalent:

- $\det A \neq 0$
- $\text{rank}(A) = n$
- A is invertible

Example

Problem. Find $\det A^T A$ given that A is defined as shown below:

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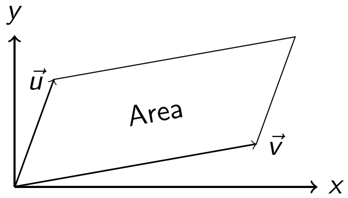
Area of a Parallelogram

Finding area given two spanning vectors



Given a vectors $\vec{u}, \vec{v}$ that span a parallelogram, the area is given by:

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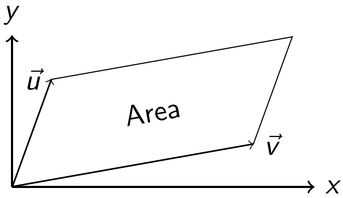

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In general, if vectors $\vec{u}, \vec{v}$ don't lie in the same plane (eg. $\vec{u}, \vec{v} \in \mathbb{R}^3$)

$$Area = ||\vec{u} \times \vec{v}|| = \left| \det \begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \right|$$

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$\therefore$ Area is equal to 2 area units.

Volume of parallelepiped

Given vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, the volume of the parallelepiped that they span up is:

$$Volume = |\vec{w} \cdot (\vec{u} \times \vec{v})| = \left| \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \right|$$

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Although the geometrical fact is useful, keep in mind the trick of converting $|\vec{w} \cdot (\vec{u} \times \vec{v})|$ to the determinant form.