Determinants

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Overview

What is a Determinant?

Definition

Determinant

In linear algebra, the determinant is a **useful value** that can be **computed** from the elements of a **square matrix**. (from `https://en.wikipedia.org/wiki/Determinant`)

For example,

- Deducing if matrix A is invertible, i.e. $\exists A^{-1}$
- Finding area of a parallelogram
- Calculating **eigenvalues** (next week)

Computing determinant

The neat property of determinants is that we can always express a determinant of matrix $n \times n$ using determinants of size $(n-1) \times (n-1)$.

Base case. 2×2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Increasing dimension. 3×3

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ d & e & f \\ j & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ j & i \end{vmatrix} + c \begin{vmatrix} d & e \\ j & h \end{vmatrix}$$

$$\begin{vmatrix} \mathbf{a} & b & c \\ \mathbf{d} & e & f \\ \mathbf{j} & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + j \begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

Determine sign of the coefficients

Question. In the example below, why is a positive but b negative?

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ d & e & f \\ j & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ j & i \end{vmatrix} + c \begin{vmatrix} d & e \\ j & h \end{vmatrix}$$

For 3×3, we have that:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

In general, the coefficient is

$$(-1)^{i+j}$$

for element in ith row and jth column

Example

Problem. Compute the determinant of A, defined as:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 0 & 6 \end{pmatrix}$$

Step 1: "Golden rule of determinant calculations: be lazy". Pick the row/column with most zeros. Why?

$$A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{6} \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 6(3-2) = 6$$

Rules and Theorems

Rules (idea)

Idea. Determinant matrix reductions are carried out to make it easier to figure out the determinant. The rules differ from elementary row operations.

Goal. Get as many zeros as possible, for then we get less terms to compute.

Rules (continued)

Constant term factorization. $\det B = c \det A$

$$\begin{vmatrix} c*1 & c*2 & c*3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = c \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Swapping two rows. $\det B = -\det A$

$$\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = - \begin{vmatrix} 4 & 5 & 6 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 7 & 8 & 9 \end{vmatrix}$$

Adding two rows/<u>columns</u>. $\det B = \det A$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \{R_2 : R_2 - R_1\} = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 0 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Theorems

Theorem. Let A be a square matrix, i.e. $n \times n$. Then:

- $\det A = \det A^T$
- $\det A^{-1} = \frac{1}{\det A}$
- $\det AB = \det A \det B$
- If two rows are equal $\implies \det A = 0$

Theorem. Given that A is a square matrix, i.e. $n \times n$, the following statements are equivalent:

- $\det A \neq 0$
- $\text{rank}(A) = n$
- A is invertible

Example

Problem. Find $\det A^T A$ given that A is defined as shown below:

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Step 1: (using known theorem): $\det(A^T A) = (\det A)^2$.

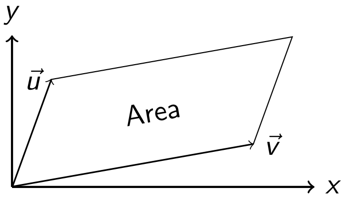
Step 2: Find $\det A$.

$$\begin{vmatrix} 1 & 1 & 2 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 4 \end{vmatrix} = -\begin{vmatrix} \mathbf{1} & 1 & 2 & 0 \\ \mathbf{0} & 1 & -1 & 1 \\ \mathbf{0} & 0 & -3 & 1 \\ \mathbf{0} & 0 & 1 & 4 \end{vmatrix} =$$

$$= -\begin{vmatrix} \mathbf{1} & -1 & 1 \\ \mathbf{0} & -3 & 1 \\ \mathbf{0} & 1 & 4 \end{vmatrix} = -\begin{vmatrix} -3 & 1 \\ 1 & 4 \end{vmatrix} = -(-3 \times 4 - 1 \times 1) = 13$$

Area of a Parallelogram

Finding area given two spanning vectors



Given a vectors $\vec{u}, \vec{v}$ that span a parallelogram, the area is given by:

$$Area = \left| det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right|$$

In general, if vectors $\vec{u}, \vec{v}$ don't lie in the same plane (eg. $\vec{u}, \vec{v} \in \mathbb{R}^3$)

$$Area = ||\vec{u} \times \vec{v}|| = \left| \det \begin{pmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \right|$$

Example

Problem. Given that a parallelogram is spanned by vectors $\vec{u} = (1, 2)$ and $\vec{v} = (3, 4)$, find its area.

Step 1: Plug vectors into a matrix:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 * 4 - 2 * 3 = -2$$

Step 2: Take the absolute value, since it's an area:

$$|-2| = 2$$

$\therefore$ Area is equal to 2 area units.

Volume of parallelepiped

Given vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, the volume of the parallelepiped that they span up is:

$$Volume = |\vec{w} \cdot (\vec{u} \times \vec{v})| = \left| \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \right|$$

Although the geometrical fact is useful, keep in mind the trick of converting $|\vec{w} \cdot (\vec{u} \times \vec{v})|$ to the determinant form.