Diagonalisations and ON-Bases

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Overview

1. What is Diagonalization?
2. Theorems
3. Orthonormal Bases
4. Exam Question

What is Diagonalization?

Definition

Diagonalization of a matrix

Consider matrix A. Let $\vec{e_1} \ldots \vec{e_n}$ be a basis of its **eigenvectors** associated with **eigenvalues** $\lambda_1 \ldots \lambda_n$. Then,

$$AP = P \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{pmatrix}$$

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We can use it to:

- Identify shape such as an ellipsoid even if was rotated.
- Compute A^{1000}.

Example from previous lesson (part B)

In Eigenvectors (slide 13), we got that T matrix had eigenvalues λ_0 and $\lambda_{0.5}$, with eigenvectors shown below:

$$\lambda_0 : \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & -0.5 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \lambda_{0.5} : \left(\begin{array}{ccc|c} -0.5 & 0 & 0 & 0 \\ -2 & 0 & 1.5 & 0 \\ 0 & 0 & -0.5 & 0 \end{array}\right)$$

$\text{span}\{(-0.25, 1, 0), (0.75, 0, 1)\}$ $\qquad$ $\text{span}\{(0, 1, 0)\}$

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Problem. Can we now express T as $T = PDP^{-1}$?

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If we can write A as $A = PDP^{-1}$, then we know that:

- $\det A = \det D$
- Both A and D have the same eigenvalues
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We can also compute high powers much easier:

$$A^k = PD^kP^{-1}$$

Example

Problem. Given P, can A be diagonalized?

$$A = \begin{pmatrix} 11 & 6 \\ 9 & -4 \end{pmatrix}, \qquad P = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

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Step 3: Confirm that $\det A = \det D$.

Ortho**N**ormal Bases

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What does this mean in the context of base change and rotation?

Example

Problem. Convert the vector $\vec{x} = (4, 3, 5)$ to base spanned by $\{\frac{1}{3}(1, -2, 2), \frac{1}{3}(2, 2, 1), \frac{1}{3}(-2, 1, 2)\}$.

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$\therefore [\vec{w}]_B = (\frac{8}{3}, \frac{19}{3}, \frac{5}{3})$

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$\therefore \lambda$ can either be 1 or -1.