Diagonalisations and ON-Bases

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Overview

What is Diagonalization?

Definition

Diagonalization of a matrix

Consider matrix A. Let $\vec{e_1} \ldots \vec{e_n}$ be a basis of its **eigenvectors** associated with **eigenvalues** $\lambda_1 \ldots \lambda_n$. Then,

$$AP = P \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{pmatrix}$$

where P is

$$P = \begin{pmatrix} | & & | \\ \vec{e_1} & \ldots & \vec{e_n} \\ | & & | \end{pmatrix}$$

We can use it to:

- Identify shape such as an ellipsoid even if was rotated.
- Compute A^{1000}.

Example from previous lesson (part B)

In Eigenvectors (slide 13), we got that T matrix had eigenvalues λ_0 and $\lambda_{0.5}$, with eigenvectors shown below:

$$\lambda_0 : \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & -0.5 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \lambda_{0.5} : \left(\begin{array}{ccc|c} -0.5 & 0 & 0 & 0 \\ -2 & 0 & 1.5 & 0 \\ 0 & 0 & -0.5 & 0 \end{array}\right)$$

$$\text{span}\{(-0.25, 1, 0), (0.75, 0, 1)\} \qquad \text{span}\{(0, 1, 0)\}$$

Problem. Can we now express T as $T = PDP^{-1}$?

Theorems

Theorems

If we can write A as $A = PDP^{-1}$, then we know that:

- $\det A = \det D$
- Both A and D have the same eigenvalues
- $\text{rank}(A) = \text{rank}(D)$

A matrix A is only diagonalizable if it has n unique eigenvectors. Furthermore, the eigenvectors should span up $\mathbb{R}^n$.

We can also compute high powers much easier:

$$A^k = PD^kP^{-1}$$

Example

Problem. Given P, can A be diagonalized?

$$A = \begin{pmatrix} 11 & 6 \\ 9 & -4 \end{pmatrix}, \qquad P = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

Step 1: Use the definition $A\vec{x} = \lambda\vec{x}$, i.e. compute $A\vec{x}$ and check that it's a multiple of $\vec{x}$.

$$\begin{pmatrix} 11 & 6 \\ 9 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 28 \\ 14 \end{pmatrix} = 14\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Longrightarrow \text{ yes}$$

$$\begin{pmatrix} 11 & 6 \\ 9 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -21 \end{pmatrix} = -7\begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \Longrightarrow \text{ yes}$$

Step 2: Find D (i.e. eigenvalues in the diagonal).

$$\begin{pmatrix} 14 & 0 \\ 0 & -7 \end{pmatrix}$$

Step 3: Confirm that $\det A = \det D$.

Ortho**N**ormal Bases

Definition

There are two terms worth distinguishing:

- **orthogonal** - 'perpendicular', i.e. $\vec{a}, \vec{b}$ are orthogonal if $\vec{a} \cdot \vec{b} = 0$
- **orthonormal** - orthogonal and normalized vectors such that $||\vec{a}|| = ||\vec{b}|| = 1$

Caution. Orthogonal matrix means that the set of column and row vectors is an orthonormal set.

Another useful property is that if a matrix is orthogonal, its inverse is the same as its transpose, i.e.

$$A^{-1} = A^T$$

What does this mean in the context of base change and rotation?

Example

Problem. Convert the vector $\vec{x} = (4, 3, 5)$ to base spanned by $\{\frac{1}{3}(1, -2, 2), \frac{1}{3}(2, 2, 1), \frac{1}{3}(-2, 1, 2)\}$.

Step 1: Use the fact that $b_i = \vec{x} \cdot \vec{v_i}$

$$\frac{1}{3}(4, 3, 5)(1, -2, 2) = \frac{8}{3}$$

$$\frac{1}{3}(4, 3, 5)(2, 2, 1) = \frac{19}{3}$$

$$\frac{1}{3}(4, 3, 5)(-2, 1, 2) = \frac{5}{3}$$

$\therefore [\vec{w}]_B = (\frac{8}{3}, \frac{19}{3}, \frac{5}{3})$

Exam Question

Exam question

Problem. Show that 1 and -1 are the only eigenvalues of orthogonal matrices.

Prerequisite.

- $x^T \cdot x$ - square of the magnitude of x (expressed using dot product).
- If A is orthogonal, then $A^{-1} = A^T$

Exam question

Problem. Show that 1 and -1 are the only eigenvalues of orthogonal matrices.

Step 1: Show that if let $A\vec{y} = x$, then $||A\vec{y}|| = ||x||$

$$x^T \cdot x = (A\vec{y})^T \cdot A\vec{y} = \vec{y}^T A^T \cdot A\vec{y} = \vec{y}^T \vec{y}$$

$\therefore$ An orthogonal matrix does not affect the magnitude

Step 2: Let $\vec{y}$ be an eigenvector with eigenvalue λ

$$||\vec{x}|| = ||A\vec{y}|| = ||\lambda\vec{y}|| = ||\lambda\vec{x}|| = |\lambda|||\vec{x}||$$

$\therefore \lambda$ can either be 1 or -1.