Eigenvalues and Eigenvectors

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Overview

What is an Eigenvalue?

Introduction problem

Problem. Given $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$ find $A^{1000}(2,1)$ using $A(1,1) = (1,1)$ and $A(1,-1) = \frac{1}{2}(1,-1)$.

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$$= x_1(1,1) = (3/2, 3/2)$$

Based on lecture notes from Olof Heden's lecture in *SF1604 Linear Algebra (in 2014)*.

Definition

Eigenvalue

The number λ is an **eigenvalue** of matrix A if

$$A\vec{x} = \lambda\vec{x}$$

for some vector $\vec{x} \neq 0$.
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We can use it to:

- find A^{1000} given A.
- perform diagonalisations

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To find corresponding eigenvectors. Solve for $\vec{v}$ (for λ_2 and λ_3):

$$(A - \lambda I)\vec{v} = \vec{0}$$

The dimension of the solution space is the geometric multiplicity.

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$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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Geometric multiplicity is 1 in both cases, since the dimension is 1.

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a) Bestäm egenvärden och egenrum till T. **(3 p)**

b) Avgör om standardmatrisen för T är diagonaliserbar. **(1 p)**

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Idea.

Step 1: Find the T matrix in the standard basis using Martin's method, i.e. $T(1,0,0) = \dots, T(0,1,0) = \dots, T(0,0,1) = \dots$.

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Step 3: Find the eigenvectors using $(A - \lambda I)\vec{v} = \vec{0}$ for each unique eigenvalue.

Exam question (continued)

Step 1: The standard matrix T is given by:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ -2 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

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$\therefore \lambda = 0, \quad \lambda = 0.5.$ Keep in mind the **double root** for λ_0.

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$$\lambda_0 : \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 0.5 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \lambda_{0.5} : \left(\begin{array}{ccc|c} -0.5 & 0 & 0 & 0 \\ -2 & 0 & 1.5 & 0 \\ 0 & 0 & -0.5 & 0 \end{array}\right)$$

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$$\text{span}\{(-0.25, 1, 0), (0.75, 0, 1)\} \qquad \text{span}\{(0, 1, 0)\}$$

Exam question (reflections)

- The eigenvectors span up a space. λ_0 had a two-dimensional eigenvector space, whereas $\lambda_{0.5}$ had one-dimensional.

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- An alternative approach would be to use the definition of eigenvalues i.e. $T\vec{a} = \lambda\vec{a}$. We would clearly see that

$$T(1,1,1) = 0 \times (1,1,1) = \overbrace{(0,0,0)}^{T(1,1,1)}$$

$$T(0,2,0) = 0.5 \times (0,2,0) = \overbrace{(0,1,0)}^{T(0,2,0)}$$

and that there is no c such that $T(0,1,1) = c \times (0,1,1) = \overbrace{(0,2,0)}^{T(0,1,1)}$

Martin's method

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Idea. Put all the vectors as rows in the matrix. Those being transformed are on the left (i.e. $\vec{v}$) and the result on the right (i.e. $T(\vec{v})$). Use row operations to get the identify matrix on LHS.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \end{array}\right) \sim \text{el row op} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 1.5 & 0 \end{array}\right)$$

If you transpose what you have on RHS, you have the standard matrix for the transformation. So, $T(1,0,0) = (0,-2,0)$, for example.