Eigenvalues and Eigenvectors

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Overview

What is an Eigenvalue?

Introduction problem

Problem. Given $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$ find $A^{1000}(2,1)$ using $A(1,1) = (1,1)$ and $A(1,-1) = \frac{1}{2}(1,-1)$.

Step 1: Find x_1, x_2 so that $(2,1) = x_1(1,1) + x_2(1,-1)$. $x_1 = \frac{3}{2}$ and $x_2 = \frac{1}{2}$

Step 2:

$$A^{1000}(2,1) = x_1 A^{1000}(1,1) + x_2 A^{1000}(1,-1) =$$

$$= x_1(1,1) + x_2 \left(\frac{1}{2}\right)^{1000} (1,-1) =$$

$$= x_1(1,1) = (3/2, 3/2)$$

Based on lecture notes from Olof Heden's lecture in *SF1604 Linear Algebra (in 2014)*.

Definition

Eigenvalue

The number λ is an **eigenvalue** of matrix A if

$$A\vec{x} = \lambda\vec{x}$$

for some vector $\vec{x} \neq 0$.
$\vec{x}$ is an **eigenvector** to A belonging to the **eigenvalue** λ.

We can use it to:

- find A^{1000} given A.
- perform diagonalisations

Finding eigenvalues and eigenvectors

Finding eigenvalues and their corresponding eigenvectors

Using the definition, we can deduce the following:
To find eigenvalues. Solve for λ (in the characteristic equation below):

$$\det(A - \lambda I) = 0$$

Eg. we may get $(2-\lambda)(3-\lambda)^2 = 0$, so the eigenvalues are λ_2 and λ_3.

The algebraic multiplicity of λ_2 and λ_3 are 1 and 2, respectively.

To find corresponding eigenvectors. Solve for $\vec{v}$ (for λ_2 and λ_3):

$$(A - \lambda I)\vec{v} = \vec{0}$$

The dimension of the solution space is the geometric multiplicity.

Example

Problem. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$$

Step 1: Solve for λ in $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) = 0$$

The roots are $\lambda_1 = 2$ and $\lambda_2 = 3$, which are also the eigenvalues of A. Both of them have **algebraic multiplicity** of 1, since we don't have any double roots, etc.

Example (continued)

Problem. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -2 \\ 0 & 3 \end{pmatrix}$$

Step 2: To find the corresponding eigenvectors, solve $(A - \lambda I)\vec{v} = \vec{0}$

<u>Case $\lambda = 2$</u>

$$\left(\begin{array}{cc|c} 0 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

x_2 has to be 0 and x_1 can be any, let's say 1.

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

<u>Case $\lambda = 3$</u>

$$\left(\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

By parametrization of $-x_1 - 2x_2 = 0$, we get:

$$\vec{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Geometric multiplicity is 1 in both cases, since the dimension is 1.

Exam question

Exam question

5. Låt $T : \mathbb{R}^3 \to \mathbb{R}^3$ vara en linjär avbildning så att:

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\2\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \text{och} \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\\0\end{bmatrix}.$$

a) Bestäm egenvärden och egenrum till T. **(3 p)**

b) Avgör om standardmatrisen för T är diagonaliserbar. **(1 p)**

From `https://www.kth.se/social/files/5858ed94f276542bb5877d62/tentor.pdf`, p. 2

Idea.

Step 1: Find the T matrix in the standard basis using Martin's method, i.e. $T(1,0,0) = \dots, T(0,1,0) = \dots, T(0,0,1) = \dots$.
This is optional but good practise.

Step 2: Find the eigenvalues using $\det(A - \lambda I) = 0$

Step 3: Find the eigenvectors using $(A - \lambda I)\vec{v} = \vec{0}$ for each unique eigenvalue.

Exam question (continued)

Step 1: The standard matrix T is given by:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ -2 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2: Find the eigenvalues using $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 0 & 0 \\ -2 & \frac{1}{2} - \lambda & \frac{3}{2} \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} \frac{1}{2} - \lambda & \frac{3}{2} \\ 0 & -\lambda \end{vmatrix} = \lambda^2(0.5 - \lambda) = 0$$

$\therefore \lambda = 0, \quad \lambda = 0.5$. Keep in mind the **double root** for λ_0.

Step 3: Find the eigenvectors using $(A - \lambda I)\vec{v} = \vec{0}$ for λ_0 and $\lambda_{0.5}$.

$$\lambda_0 : \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 0.5 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \lambda_{0.5} : \left(\begin{array}{ccc|c} -0.5 & 0 & 0 & 0 \\ -2 & 0 & 1.5 & 0 \\ 0 & 0 & -0.5 & 0 \end{array}\right)$$

Exam question (continued)

Step 3: Find the eigenvectors using $(A - \lambda I)\vec{v} = \vec{0}$ for λ_0 and $\lambda_{0.5}$.

$$\lambda_0 : \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & -0.5 & 1.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \lambda_{0.5} : \left(\begin{array}{ccc|c} -0.5 & 0 & 0 & 0 \\ -2 & 0 & 1.5 & 0 \\ 0 & 0 & -0.5 & 0 \end{array}\right)$$

$$\text{span}\{(-0.25, 1, 0), (0.75, 0, 1)\} \qquad \text{span}\{(0, 1, 0)\}$$

Exam question (reflections)

- The eigenvectors span up a space. λ_0 had a two-dimensional eigenvector space, whereas $\lambda_{0.5}$ had one-dimensional.
- An alternative approach would be to use the definition of eigenvalues i.e. $T\vec{a} = \lambda\vec{a}$. We would clearly see that

$$T(1,1,1) = 0 \times (1,1,1) = \overbrace{(0,0,0)}^{T(1,1,1)}$$

$$T(0,2,0) = 0.5 \times (0,2,0) = \overbrace{(0,1,0)}^{T(0,2,0)}$$

and that there is no c such that $T(0,1,1) = c \times (0,1,1) = \overbrace{(0,2,0)}^{T(0,1,1)}$

Martin's method

Martin's method

Idea. Put all the vectors as rows in the matrix. Those being transformed are on the left (i.e. $\vec{v}$) and the result on the right (i.e. $T(\vec{v})$). Use row operations to get the identify matrix on LHS.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \end{array}\right) \sim \text{el row op} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 1.5 & 0 \end{array}\right)$$

If you transpose what you have on RHS, you have the standard matrix for the transformation. So, $T(1,0,0) = (0,-2,0)$, for example.