ON-Bases and Least Square Method

Artem Los
(arteml@kth.se)

February 21th, 2017

Overview

Projections onto Subspaces

Projecting a vector onto subspace

Projection of vector $\vec{b}$ onto a plane π is the **best approximation** of b in π, defined as:

$$\text{proj}_\pi = \frac{\vec{b} \cdot \vec{v_1}}{||\vec{v_1}||^2}\vec{v_1} + \frac{\vec{b} \cdot \vec{v_2}}{||\vec{v_2}||^2}\vec{v_2}$$

Projecting a vector onto subspace

Projection of vector $\vec{b}$ onto a plane π is the **best approximation** of b in π, defined as:

$$\text{proj}_\pi = \frac{\vec{b} \cdot \vec{v_1}}{||\vec{v_1}||^2}\vec{v_1} + \frac{\vec{b} \cdot \vec{v_2}}{||\vec{v_2}||^2}\vec{v_2}$$

We've assumed that $\pi = \text{span}\{\vec{v_1}, \vec{v_2}\}$. The same pattern is applied to hyper planes, etc.

Example

Problem. Find the projection of $\vec{x} = (2, 3, 5, 6)$ onto $\pi = s(1, -1, -1, 1) + t(1, 2, 1, 2) \quad : \quad s, t \in \mathbb{R}$

Example

Problem. Find the projection of $\vec{x} = (2, 3, 5, 6)$ onto
$\pi = s(1, -1, -1, 1) + t(1, 2, 1, 2) \quad : \quad s, t \in \mathbb{R}$

Step 1: Find $\text{proj}_\pi(\vec{x})$ using the definition

$$\text{proj}_\pi = \frac{\vec{b} \cdot \vec{v_1}}{||\vec{v_1}||^2}\vec{v_1} + \frac{\vec{b} \cdot \vec{v_2}}{||\vec{v_2}||^2}\vec{v_2} =$$

Example

Problem. Find the projection of $\vec{x} = (2, 3, 5, 6)$ onto
$\pi = s(1, -1, -1, 1) + t(1, 2, 1, 2) \quad : \quad s, t \in \mathbb{R}$

Step 1: Find $\text{proj}_\pi(\vec{x})$ using the definition

$$\text{proj}_\pi = \frac{\vec{b} \cdot \vec{v_1}}{||\vec{v_1}||^2}\vec{v_1} + \frac{\vec{b} \cdot \vec{v_2}}{||\vec{v_2}||^2}\vec{v_2} =$$

$$\text{proj}_\pi \vec{x} = \frac{(2, 3, 5, 6) \cdot (1, -1, -1, 1)}{||(1, -1, -1, 1)||^2}(1, -1, -1, 1) +$$
$$+ \frac{(2, 3, 5, 6) \cdot (1, 2, 1, 2)}{||(1, 2, 1, 2)||^2}(1, 2, 1, 2) =$$

Example

Problem. Find the projection of $\vec{x} = (2,3,5,6)$ onto $\pi = s(1,-1,-1,1) + t(1,2,1,2) \quad : \quad s,t \in \mathbb{R}$

Step 1: Find $\text{proj}_\pi(\vec{x})$ using the definition

$$\text{proj}_\pi = \frac{\vec{b}\cdot\vec{v_1}}{||\vec{v_1}||^2}\vec{v_1} + \frac{\vec{b}\cdot\vec{v_2}}{||\vec{v_2}||^2}\vec{v_2} =$$

$$\text{proj}_\pi\vec{x} = \frac{(2,3,5,6)\cdot(1,-1,-1,1)}{||(1,-1,-1,1)||^2}(1,-1,-1,1) +$$

$$+\frac{(2,3,5,6)\cdot(1,2,1,2)}{||(1,2,1,2)||^2}(1,2,1,2) =$$

$$= \left(\frac{5}{2}, 5, \frac{5}{2}, 5\right)$$

Orthogonal Complement

Orthogonal complement

Relationship with row space

For all matrices A, the **null space** of A is an orthogonal complement to the **row space** of A.

Orthogonal complement

Relationship with row space

For all matrices A, the **null space** of A is an orthogonal complement to the **row space** of A.

How about A^T?

Example

Problem. Find the basis of the orthogonal complement of $S = \text{span}\{(1, 2, -1)\}$, i.e. $S^{\perp}$.

Example

Problem. Find the basis of the orthogonal complement of $S = \text{span}\{(1, 2, -1)\}$, i.e. $S^\perp$.

Step 1: Transform into a plane.

Example

Problem. Find the basis of the orthogonal complement of $S = \text{span}\{(1, 2, -1)\}$, i.e. $S^{\perp}$.

Step 1: Transform into a plane.
$S = \text{span}\{(1, 2, -1)\}$ means the equation of the plane is $x_1 + 2x_2 - x_3$.

Example

Problem. Find the basis of the orthogonal complement of $S = \text{span}\{(1, 2, -1)\}$, i.e. $S^\perp$.

Step 1: Transform into a plane.
$S = \text{span}\{(1, 2, -1)\}$ means the equation of the plane is $x_1 + 2x_2 - x_3$.

Step 2: Insert this into a matrix (interpret as row space)
Since we want to find the null space, we solve $x_1 + 2x_2 - x_3 = 0$.

Example

Problem. Find the basis of the orthogonal complement of $S = \text{span}\{(1,2,-1)\}$, i.e. $S^{\perp}$.

Step 1: Transform into a plane.
$S = \text{span}\{(1,2,-1)\}$ means the equation of the plane is $x_1 + 2x_2 - x_3$.

Step 2: Insert this into a matrix (interpret as row space)
Since we want to find the null space, we solve $x_1 + 2x_2 - x_3 = 0$.

Step 3: Use parametrisation to get the solution space (here it's null space)

$$s(-2,1,0) + t(1,0,1) \qquad s,t \in \mathbb{R}$$

Gram-Schmidt Method

Find orthonormal basis using Gram-Schmidt Method

Goal. We want to find an **orthonormal basis** given $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$. We will see later that this is useful.

Find orthonormal basis using Gram-Schmidt Method

Goal. We want to find an **orthonormal basis** given $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$. We will see later that this is useful.

Algorithm.

$$
\begin{aligned}
&\text{Step1}: \vec{u_1} = \vec{v_1} \\
&\text{Step2}: \vec{u_2} = \vec{v_2} - \frac{\vec{v_2}\cdot\vec{u_1}}{||\vec{u_1}||^2}\vec{u_1} \\
&\text{Step2}: \vec{u_3} = \vec{v_3} - \frac{\vec{v_3}\cdot\vec{u_1}}{||\vec{u_1}||^2}\vec{u_1} - \frac{\vec{v_3}\cdot\vec{u_2}}{||\vec{u_2}||^2}\vec{u_2} \\
&\vdots \\
&\text{r times}
\end{aligned}
$$

Example

Problem. We are given a basis spanned by $(1,1,1),(1,1,0),(1,0,0)$. Find an orthonormal basis.

Example

Problem. We are given a basis spanned by $(1, 1, 1), (1, 1, 0), (1, 0, 0)$. Find an orthonormal basis.

Step 1: Find an orthogonal basis using Gram-Schmidt process.

Example

Problem. We are given a basis spanned by $(1,1,1),(1,1,0),(1,0,0)$. Find an orthonormal basis.

Step 1: Find an orthogonal basis using Gram-Schmidt process.

Step 2: Normalize the new vectors to get an orthonormal basis.

Example

Problem. We are given a basis spanned by $(1,1,1), (1,1,0), (1,0,0)$. Find an orthonormal basis.

Step 1: Find an orthogonal basis using Gram-Schmidt process.

Step 2: Normalize the new vectors to get an orthonormal basis.

Example in Python

Least Square Regression

Definition

Least Square Method

The method of least squares is a standard approach in regression analysis to the **approximate solution** of **overdetermined** systems, i.e., sets of equations in which there are **more equations than unknowns**.

(From https://en.wikipedia.org/wiki/Least_squares)

Definition

Least Square Method

The method of least squares is a standard approach in regression analysis to the **approximate solution** of **overdetermined** systems, i.e., sets of equations in which there are **more equations than unknowns**.

(From https://en.wikipedia.org/wiki/Least_squares)

Examples of usage:

- Find the equation of straight line going through a set of points (eg. from an experiment).
- Fitting a curve to set of points

The trick of solving overdetermined systems

Let A be the matrix with more equations than unknowns (i.e. overdetermined). Then, to minimize $||A\vec{x} - \vec{b}||$ is the same as solving $A^T A\vec{x} = A^T \vec{b}$. So,

$$A^T A\vec{x} = A^T \vec{b} \qquad \Longrightarrow \qquad \vec{x} = (A^T A)^{-1} A^T \vec{b}$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$$

Step 1: Find A

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix}$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$$

Step 1: Find A

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix}$$

Step 2: Find $A^T A$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 34 \end{pmatrix}$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$$

Step 3: Find $A^T\vec{b}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$$

Step 3: Find $A^T \vec{b}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Step 4: Solve $A^T A = A^T \vec{b}$

$$\begin{pmatrix} 5 & 10 \\ 10 & 34 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2,1),(-1,1),(0,2),(1,3),(2,-2)$$

Step 3: Find $A^T\vec{b}$

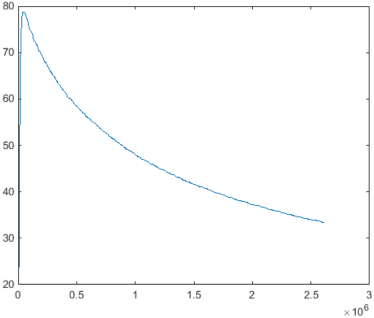
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

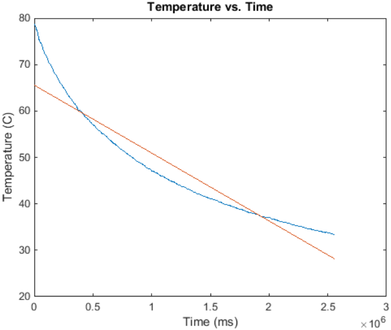
Step 4: Solve $A^T A = A^T\vec{b}$

$$\begin{pmatrix} 5 & 10 \\ 10 & 34 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Example in Python

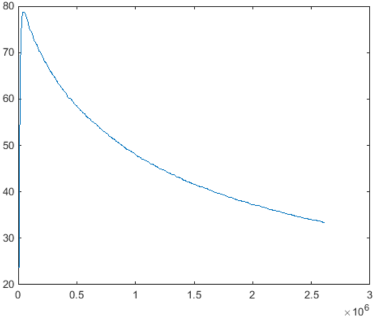
Question. Find a relationship between the temperature of Artem's tea cup and time. How long time is the tea still drinkable?

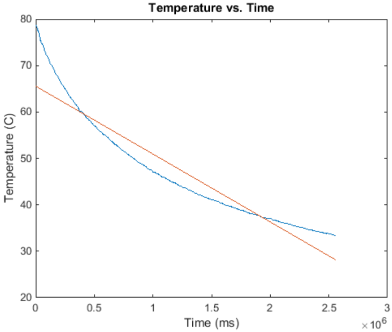




Application for Least Squared

Question. Find a relationship between the temperature of Artem's tea cup and time. How long time is the tea still drinkable?





Data can be found here