

ON-Bases and Least Square Method

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Projections onto Subspaces

Projecting a vector onto subspace

Projection of vector \vec{b} onto a plane π is the **best approximation** of b in π , defined as:

$$\text{proj}_{\pi} = \frac{\vec{b} \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\vec{b} \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2$$

We've assumed that $\pi = \text{span}\{\vec{v}_1, \vec{v}_2\}$. The same pattern is applied to hyper planes, etc.

Example

Problem. Find the projection of $\vec{x} = (2, 3, 5, 6)$ onto $\pi = s(1, -1, -1, 1) + t(1, 2, 1, 2) \quad : \quad s, t \in \mathbb{R}$

Step 1: Find $\text{proj}_\pi(\vec{x})$ using the definition

$$\begin{aligned}\text{proj}_\pi &= \frac{\vec{b} \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\vec{b} \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2 = \\ \text{proj}_\pi \vec{x} &= \frac{(2, 3, 5, 6) \cdot (1, -1, -1, 1)}{\|(1, -1, -1, 1)\|^2} (1, -1, -1, 1) + \\ &+ \frac{(2, 3, 5, 6) \cdot (1, 2, 1, 2)}{\|(1, 2, 1, 2)\|^2} (1, 2, 1, 2) = \\ &= \left(\frac{5}{2}, 5, \frac{5}{2}, 5 \right)\end{aligned}$$

Orthogonal Complement

Relationship with row space

For all matrices A , the **null space** of A is an orthogonal complement to the **row space** of A .

How about A^T ?

Example

Problem. Find the basis of the orthogonal complement of $S = \text{span}\{(1, 2, -1)\}$, i.e. S^\perp .

Step 1: Transform into a plane.

$S = \text{span}\{(1, 2, -1)\}$ means the equation of the plane is $x_1 + 2x_2 - x_3$.

Step 2: Insert this into a matrix (interpret as row space)

Since we want to find the null space, we solve $x_1 + 2x_2 - x_3 = 0$.

Step 3: Use parametrisation to get the solution space (here it's null space)

$$s(-2, 1, 0) + t(1, 0, 1) \quad s, t \in \mathbb{R}$$

Gram-Schmidt Method

Find orthonormal basis using Gram-Schmidt Method

Goal. We want to find an **orthonormal basis** given $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$. We will see later that this is useful.

Algorithm.

$$\text{Step1 : } \vec{u}_1 = \vec{v}_1$$

$$\text{Step2 : } \vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\|\vec{u}_1\|^2} \vec{u}_1$$

$$\text{Step2 : } \vec{u}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{u}_1}{\|\vec{u}_1\|^2} \vec{u}_1 - \frac{\vec{v}_3 \cdot \vec{u}_2}{\|\vec{u}_2\|^2} \vec{u}_2$$

⋮

r times

Example

Problem. We are given a basis spanned by $(1, 1, 1)$, $(1, 1, 0)$, $(1, 0, 0)$. Find an orthonormal basis.

Step 1: Find an orthogonal basis using Gram-Schmidt process.

Step 2: Normalize the new vectors to get an orthonormal basis.

[Example in Python](#)

Least Square Regression

Least Square Method

The method of least squares is a standard approach in regression analysis to the **approximate solution** of **overdetermined** systems, i.e., sets of equations in which there are **more equations than unknowns**.

(From https://en.wikipedia.org/wiki/Least_squares)

Examples of usage:

- Find the equation of straight line going through a set of points (eg. from an experiment).
- Fitting a curve to set of points

The trick of solving overdetermined systems

Let A be the matrix with more equations than unknowns (i.e. overdetermined). Then, to minimize $\|A\vec{x} - \vec{b}\|$ is the same as solving $A^T A\vec{x} = A^T \vec{b}$. So,

$$A^T A\vec{x} = A^T \vec{b} \quad \implies \quad \vec{x} = (A^T A)^{-1} A^T \vec{b}$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$$

Step 1: Find A

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix}$$

Step 2: Find $A^T A$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 34 \end{pmatrix}$$

Example

Problem. We want to fit $y = a + bt^2$ and we are given five data points:

$$(-2, 1), (-1, 1), (0, 2), (1, 3), (2, -2)$$

Step 3: Find $A^T \vec{b}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

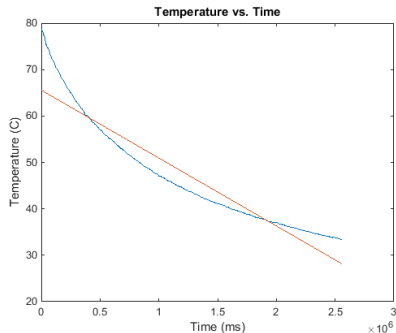
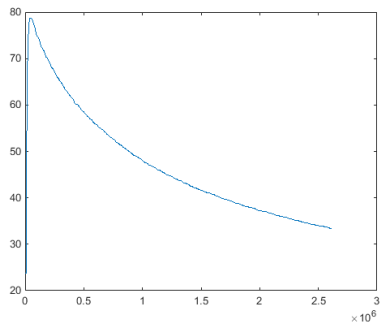
Step 4: Solve $A^T A = A^T \vec{b}$

$$\begin{pmatrix} 5 & 10 \\ 10 & 34 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Example in Python

Application for Least Squared

Question. Find a relationship between the temperature of Artem's tea cup and time. How long time is the tea still drinkable?



Data can be found [here](#)