Spectral Theorem and Quadratic Forms

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Overview

1. Symmetric Matrices
2. Quadratic Form
3. Recommended reading (review)

Symmetric Matrices

Definition

Symmetric Matrix

A symmetric matrix is one that satisfies $A^T = A$.

As we will see later, **quadratic form** is one of the ways of exploiting **diagonalization symmetric matrices**.

Theorems

When we want to perform 'orthogonal diagonalization', we note that:

- The eigenvectors of a symmetric matrix (with unique eigenvalues) are orthogonal.
- For any other case, we can always use Gram-Schmidt method to find an orthonormal basis.

Example

Problem. Find P, D in $A = PDP^T$, given that:

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

Step 1: Find the eigenvalues of A and corresponding eigenvectors

$$A(-1,1) = 3(-1,1) \qquad A(1,1) = -1(1,1)$$

Step 2: Normalize the vectors

$$\vec{u_1} = \frac{1}{\sqrt{2}}(-1,1) \qquad \vec{u_2} = \frac{1}{\sqrt{2}}(1,1)$$

We know from a theorem that $(-1,1)$ and $(1,1)$ are orthogonal since A is symmetric. It follows that $\vec{u_1}, \vec{u_2}$ form an ON-base and P and orthogonal matrix.

Step 3: Deduce P, D

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Quadratic Form

Definition

Motivation. Sometimes, we have equations of shapes that are transformed in some way (eg. rotated ellipse). However, we want to be able to deduce certain information (eg. major/minor of an ellipse). Then, we want to change the coordinate system in such a way that the ellipse is depicted in the 'normal form', with centre in the origin.

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = ax_1^2 + bx_1x_2 + cx^2$$

Note: it can also have many more terms than displayed above.
The point is, we want to diagonalize Q to get rid of non-quadratic terms, i.e. here we want $b = 0$.

Classification of quadratic forms

- $Q(\vec{x})$ is positive definite if $Q(\vec{x}) > 0 \qquad \forall x \neq 0$
- $Q(\vec{x})$ is positive semi-definite if $Q(\vec{x}) \geq 0 \qquad \forall x$
- $Q(\vec{x})$ is positive indefinite if $Q(\vec{x}) > 0$ for some $\vec{x}$ and $Q(\vec{x}) < 0$ for some $\vec{x}$

Other forms (eg. negative definite) follow from these definitions.

Example (step-by-step, no theory)

Problem. Express $Q(\vec{x}) = x_1^2 - 3x_1x_2 + x_2^2$ with only quadratic terms.

Step 1: Find A such that $Q(\vec{x}) = \vec{x}^T A\vec{x}$

$$A = \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}$$

Step 2: Find eigenvalues of A.

$$\lambda = 2.5 \qquad \lambda = -0.5$$

Step 3: Construct D matrix

$$D = \begin{pmatrix} 2.5 & 0 \\ 0 & -0.5 \end{pmatrix}$$

$$\therefore Q(\vec{y}) = 2.5y_1^2 - 0.5y_2^2$$

Example (explanation)

Problem. Express $Q(\vec{x}) = x_1^2 - 3x_1x_2 + x_2^2$ with only quadratic terms.

1: We are given $Q(\vec{x}) = \vec{x}^T A\vec{x}$.
2: Let $\vec{x} = P\vec{y}$
3: $Q(\vec{x}) = \vec{x}^T A\vec{x} = (P\vec{y})^T A(P\vec{y}) = \vec{y}^T(P^T AP)\vec{y}$.

But, what is $P^T AP$? Recall $AP = PD$, so $D = P^T AP$.

P is now our conversion matrix to go from one coordinate system to another, i.e. from x_1, x_2 to y_1, y_2 and vice versa. It's defined in (2).

Task: Express $(3, 2)_X$ in Y coordinates

Example

Problem. Classify $A = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$. In other words, classify $4x_2^2 - 4x_1x_2 + 4x_2^2$.

Step 1: Find D by finding the eigenvectors of A.

$$\lambda = 6 \qquad \lambda = 2$$
$$\therefore D = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

Step 2: Classify the matrix
Since for all $\vec{y} \neq 0$, $Q(\vec{y}) = 6y_1^2 + 2y_2^2 > 0$, A is positive definite.

Recommended reading (review)

- Eigenvalues
- Diagonalization (general technique)